SEISMIC RESPONSES OF POST-YIELD HARDENING SINGLE-DEGREE-OF-FREEDOM SYSTEMS INCORPORATING HIGH-STRENGTH ELASTIC MATERIAL

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ABSTRACT
It is widely accepted that ductility design improves the seismic capacity of structures worldwide. Nevertheless, inelastic deformation allows serious damage to occur in structures. Previous studies have shown that a certain level of post-yield stiffness may reduce both the peak displacement and residual deformation of a structure. In recent years, several high-strength elastic materials, such as fiber-reinforced polymer (FRP) and high-strength steel bars, have been developed. Application of these materials can easily provide a structure with a much higher and more stable post-yield stiffness. Many materials, members and structures that incorporate both high-strength elastic materials and conventional materials show significant post-yield hardening (PYH) behaviors. The significant post-yield stiffness of PYH structures can help effectively reduce both peak and residual deformations, providing a choice when designing resilient structures. However, the findings of previous studies of structures with elastic-perfectly plastic (EPP) behavior or small post-yield stiffness may not be accurate for PYH structures. The post-yield stiffness of a structure must be considered an important primary structural parameter, in addition to initial stiffness, yielding strength and ductility. In this paper, extensive time-history and statistical analyses are carried out for PYH single-degree-of-freedom (SDOF) systems. The mean values and coefficients of variation of the peak displacement and residual deformation are obtained and discussed. A new $R$-$\mu$-$T$-$\alpha$ relationship and damage index for PYH structures are proposed. A theoretical model for the calculation of residual deformation is also established. These models provide a basis for developing the appropriate seismic design and performance evaluation procedures for PYH structures.

KEYWORDS
high-strength elastic materials, post-yield stiffness, single degree of freedom, peak displacement, residual deformation, damage index

1. INTRODUCTION
The seismic design methodology employed in major seismic codes emphasizes the importance of structural ductility, which allows the structure to sustain significant inelastic deformation during earthquakes without collapsing.\textsuperscript{1} Nevertheless, the inelastic deformation of structural components allows serious damage to occur in the structure and may cause considerable economic losses. In the framework of performance-based earthquake engineering, the use of energy dissipation devices specially designed to address inelastic deformation and energy dissipation provides an effective means of protecting the primary structure from excessive seismic damage.\textsuperscript{2} In addition to increasing the energy dissipation capacity of a structure, some of these devices can also increase the overall post-yield stiffness, which is an important aspect for the seismic damage control of the structure.\textsuperscript{3} Previous studies have shown that a certain level of post-yield stiffness may result in smaller peak displacements of a structure.\textsuperscript{3, 7} More importantly, previous studies have shown that post-yield stiffness is the most important factor affecting the residual deformation of a single-degree-of-freedom (SDOF) system.\textsuperscript{8-11} Pettinga et al.\textsuperscript{12} proposed to increase the post-yield stiffness to mitigate the residual deformation of buildings. Residual deformation is an important measure for evaluating the seismic performance of a structure in terms of recovery time.\textsuperscript{13-15}

1.1 Novel post-yield hardening (PYH) behaviors from advanced materials for resilient structure development
While structures made of conventional materials may exhibit some post-yield stiffness (usually less than 10% of the initial stiffness), the application of new materials with a high strength and superior elastic properties can easily be used to develop structures with a much higher and more stable post-yield stiffness. These high-strength elastic materials, such as fiber-reinforced polymer (FRP)\textsuperscript{16-32} and high-strength steel bars,\textsuperscript{33, 34} exhibit little or
approximately no plastic behavior and have been recently used to fabricate many novel materials, members and structures. Compared with the components made of only conventional materials (steel and concrete), increasingly more materials, members and structures that incorporate both high-strength elastic materials and conventional materials show significant post-yield hardening (PYH) skeleton curves (Figure 1). A variety of such innovative components are summarized in Table 1 [Refs. 16-34 cited in Table 1]. These materials, members and structures can be called PYH materials, members and structures, respectively. Additionally, from the current point of view, it is relatively difficult to construct an entire structure with a large post-yield stiffness that is composed of many elements because, for example, of the influence of second-order effects. However, it can be easy to achieve a large post-yield stiffness for some low structures, like a bridge pier or a low-rise frame structure.

As shown in Figure 2, when a structure like a bridge pier is subjected to seismic loading, if the structure is nearly elastic-perfectly plastic (EPP), it has good ductility but serious damage will happen. If the structure is elastic, no damage will happen, but the structure is very hard and uneconomical to realize. If the structure is PYH, the significant post-yield stiffness can help effectively reduce both peak and residual deformations compared with the EPP structure, thus providing a choice when designing resilient structures. The novel PYH behaviors from advanced materials can be considered as the transition from EPP behavior to elastic behavior. In other words, elastic behavior and EPP behavior are the upper and lower bounds of the PYH behaviors, respectively. However, most existing structural analysis theories, including the seismic design methods, were developed based on elastic and EPP behaviors, and theories on PYH behaviors need to be improved.

(a) Comparison of a conventional reinforced concrete (RC) beam and a FRP-strengthened RC beam

(b) Comparison of a conventional RC column and an unbonded bar-RC column

Figure 1. Comparison of conventional and PYH components.

Figure 2. A bridge pier with different structural response characteristics subjected to seismic loading.
2. DYNAMIC TIME-HISTORY ANALYSIS

2.1 Structural models

Extensive nonlinear dynamic analyses are carried out using a custom Fortran code to evaluate the seismic responses of PYH SDOF systems. The correctness of the code has been verified in previous studies. The lumped mass of the SDOF systems is 1 kg, and the viscous damping ratio is 0.05. First, the hysteretic behavior of a PYH SDOF system should be assumed because the PYH behavior is only related to the skeleton curve. The skeleton curve of a PYH SDOF system can be understood as the combination of an EPP model and a linear elastic model in parallel because, in most of the cases shown in Table 1, the high-strength material and conventional materials (steel

<table>
<thead>
<tr>
<th>Levels</th>
<th>Types</th>
<th>Behaviors</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>Steel-fiber reinforced polymer composite bar (SFCB)</td>
<td>Axial tensile</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>FRP-confined concrete</td>
<td>Axial compressive</td>
<td>17, 18</td>
</tr>
<tr>
<td></td>
<td>Steel-concrete-FRP-concrete (SCFC) column</td>
<td>Axial compressive</td>
<td>19, 20</td>
</tr>
<tr>
<td></td>
<td>FRP-strengthened aluminum alloy short column</td>
<td>Axial compressive</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>FRP-strengthened steel column</td>
<td>Axial compressive</td>
<td>22-24</td>
</tr>
<tr>
<td>Member</td>
<td>FRP-strengthened RC beam</td>
<td>Flexural behavior</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>SFCB-reinforced beam</td>
<td>Flexural behavior</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>Hybrid steel and FRP bar reinforced beam</td>
<td>Flexural behavior</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>FRP-strengthened aluminum alloy beam</td>
<td>Flexural behavior</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>SFCB-reinforced column</td>
<td>Flexural behavior under axial compressive force</td>
<td>29, 30</td>
</tr>
<tr>
<td></td>
<td>High-strength unbonded bar-reinforced column</td>
<td>Flexural behavior under axial compressive force</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>Concrete frame (1-story 1-bay) with FRP-reinforced columns</td>
<td>Lateral behavior</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>Concrete frame (3-story 2-bay) with high-strength steel bars</td>
<td>Lateral behavior</td>
<td>34</td>
</tr>
</tbody>
</table>

1.2 Research significance: using post-yielding stiffness as the design parameter of structures

Since the 1960s, many research works have been carried out, resulting in several different formulas for the $R-\mu_p-T$ relationship, where $R$ is the strength reduction factor, $\mu_p$ is the ductility factor and $T$ is the period of vibration of the structure. Velosos and Newmark first proposed the famous equal-energy criterion for short-period structures and the equal-displacement criterion for long-period structures. The elastoplastic response spectra have also been established for seismic design and evaluation. All these efforts laid the foundation for the ductility-based seismic design and evaluation methodology that is widely adopted in the current seismic codes. However, the findings and conclusions of previous studies on structures with EPP behavior or small post-yield stiffness may not be accurate for other types of structural systems. Examples include the studies in which the $R-\mu_p-T$ relationship was used for structural systems with substantial strength degradation because of the failure of some secondary structural elements. Similarly, these findings are not readily applicable to PYH structures. Although many studies have discussed the influence of post-yield stiffness on the $R-\mu_p-T$ relationship, the post-yield stiffness ratio $\alpha$ (the ratio of post-yield to initial stiffness) was usually taken as a trivial factor rather than a primary structural parameter that may significantly influence the structural response. This approach was primarily used because (1) $\alpha$ is usually very small in conventional structures and (2) it is not feasible to adjust $\alpha$ for conventional structures, so it cannot be taken as a design parameter. PYH structures are very different from conventional structures in terms of these two aspects. The post-yield stiffness ratio of PYH structures can approach unity and is adjustable by the appropriate combination of high-strength elastic materials and conventional ductile materials. In addition, when addressing the post-yield stiffness, previous studies simply increased $\alpha$ in the hysteretic model, whereas PYH structures are better represented by the combination of a conventional hysteretic model and a fully elastic model in parallel. This result would lead to a significant difference in the residual deformation (Figure 3a). Considering the limitations of existing $R-\mu_p-T$ models in addressing the above issues, it is necessary to establish a more comprehensive $R-\mu_p-T-\alpha$ relationship that is more appropriate for PYH structures.

In this paper, an extensive dynamic time-history analysis was carried out for SDOF PYH systems. The mean values and coefficients of variation of the peak displacement and residual deformation were obtained and discussed. Based on the numerical analysis results, a $R-\mu_p-T-\alpha$ model called the straight-line criterion that explicitly takes the post-yield stiffness as a primary structural parameter is proposed for the PYH structures. A theoretical model for the quantitative calculation of residual deformation is also established. These models provide a basis for developing appropriate seismic design and performance evaluation procedures for PYH structures.
and concrete) in an element are collaborative in bearing the load. For example, in a high-strength unbonded bar-reinforced column in which high-strength unbonded bars are installed in an RC column, the high-strength unbonded bars and the conventional steel reinforcement provide the tensile resistance together. Therefore, in this study, the hysteresis model of a PYH SDOF system is assumed as the combination of the hysteresis model of a conventional structure and a linear elastic model in parallel, as shown in Figure 3a. Furthermore, the classical Takeda degrading stiffness (TKS) model was chosen to model the hysteretic behavior of conventional structures, as shown in Figure 3a, middle. The unloading stiffness degradation is described by Equation (1):

$$K_u = K_0 \left[ \frac{d_m}{d_y} \right]^\gamma$$

(1)

where $K_0$ and $K_u$ are the initial stiffness and unloading stiffness of the corresponding TKS model, respectively. $d_m$ is the peak displacement, and $d_y$ is the yield displacement. $\gamma$ is the unloading stiffness reduction factor and is set to 0.4 here. The initial stiffness $K_e$ is given in Equation (2). The post-yield stiffness ratio $\alpha$ is defined in Equation (3) (Figure 3a). Typical skeleton curves of PYH SDOF systems with different post-yield stiffness ratios are given in Figure 3b.

$$K_e = m \left( \frac{2\pi}{T} \right)^2$$

(2)

where $m$ is the lumped mass and $T$ is the initial period corresponding to the initial stiffness of the SDOF system.

$$\alpha = \frac{K_s}{K_e} = \frac{0}{1+\theta}$$

(3)

where $K_s$ is the post-yield stiffness of the SDOF system.

2.2. Input ground motions

A total of 244 horizontal components, including 22 sets of ground motions recommended by the FEMA 695 report and 100 sets of ground motions selected from the Next Generation Attenuation (NGA) Database of the Pacific Earthquake Engineering Research Center, US, were selected for the ground motion input. The latter 100 sets of ground motions were selected according to the following rules: (1) the earthquake magnitude is higher than 6.5; (2) the fault distance is greater than 10 km; (3) the peak ground acceleration (PGA) is greater than 0.04 g, and the peak ground velocity (PGV) is greater than 5 cm/s; (4) the average shear wave velocity in the 30-m soil layer is between 200-500 m/s; and (5) the recording instrument is located in a free field or on the ground floor of a low-rise
building. Selected ground motion records are shown in Table 2. The pseudo-acceleration spectra of the 244 records when they are scaled to have a PGA of 0.4 g are depicted in Figure 4. In Figure 4, the mean spectrum and design spectrum from ASCE/SEI 7-10 ($S_{DS}$ and $S_{DI}$, the two critical parameters for determining the design response spectrum, were chosen as 1.0 and 0.5, respectively) are also depicted.

![Figure 4. Elastic response spectra of the used records.](image)

Table 2. Earthquake ground motion records used in this study.

<table>
<thead>
<tr>
<th>No.</th>
<th>Magnitude</th>
<th>Year</th>
<th>Earthquake</th>
<th>Number of records</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.36</td>
<td>1952</td>
<td>Kern County</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>6.61</td>
<td>1971</td>
<td>San Fernando</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>6.50</td>
<td>1976</td>
<td>Friuli, Italy</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>7.35</td>
<td>1978</td>
<td>Tabas, Iran</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>7.54</td>
<td>1979</td>
<td>St Elias, Alaska</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>6.53</td>
<td>1979</td>
<td>Imperial Valley</td>
<td>14</td>
</tr>
<tr>
<td>7</td>
<td>6.54</td>
<td>1987</td>
<td>Superstition Hills</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>6.93</td>
<td>1989</td>
<td>Loma Prieta</td>
<td>34</td>
</tr>
<tr>
<td>9</td>
<td>7.01</td>
<td>1992</td>
<td>Cape Mendocino</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>7.28</td>
<td>1992</td>
<td>Landers</td>
<td>34</td>
</tr>
<tr>
<td>11</td>
<td>6.69</td>
<td>1994</td>
<td>Northridge</td>
<td>30</td>
</tr>
<tr>
<td>12</td>
<td>6.90</td>
<td>1995</td>
<td>Kobe, Japan</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>7.20</td>
<td>1995</td>
<td>Gulf of Aqaba</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>7.51</td>
<td>1999</td>
<td>Kocaeli, Turkey</td>
<td>20</td>
</tr>
<tr>
<td>15</td>
<td>7.62</td>
<td>1999</td>
<td>Chi-Chi, Taiwan</td>
<td>20</td>
</tr>
<tr>
<td>16</td>
<td>7.13</td>
<td>1999</td>
<td>Hector Mine</td>
<td>36</td>
</tr>
<tr>
<td>17</td>
<td>7.14</td>
<td>1999</td>
<td>Duzce, Turkey</td>
<td>4</td>
</tr>
<tr>
<td>18</td>
<td>6.6</td>
<td>1971</td>
<td>San Fernando</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>7.4</td>
<td>1990</td>
<td>Manjil, Iran</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td>244</td>
</tr>
</tbody>
</table>

2.3. Analysis process

For the existing $R$-$\mu_p$-$T$ relationship based on EPP SDOF systems, the ductility coefficient $\mu_p$ and strength reduction factor $R$ are defined in Equations (4) and (5), respectively.

$$\mu_p = \frac{d_y}{d_p} \tag{4}$$

where $d_y$ is the yield displacement and $d_p$ is the peak displacement.

$$R = \frac{F_y}{F_e} \tag{5}$$

where $F_y$ is the yield load resistance and $F_e$ is the peak load response of the corresponding elastic SDOF system. As
shown in Figure 5, to distinguish the EPP and PYH structures, the deformation coefficient of the PYH structure is defined as follows:

$$\mu_s = \frac{d_s}{d_p}$$

(6)

where $d_s$ is the peak displacement response of the PYH SDOF system. Note that $d_s = d_p$ and $\mu_s = \mu_p$ when the post-yield stiffness equals zero. Similarly, the strength reduction factor of PYH structures (Figure 5) can also be defined as Equation (5). Then, the peak load capacity response $F_s$ in Figure 5 can be expressed as:

$$F_s = \frac{F_p}{R} + (\mu_s - 1)d_s = \frac{F_p}{R}[1 + \alpha(\mu_s - 1)]$$

(7)

For existing research on the $R-\mu_p-T$ relationship, one parameter was usually fixed and then the relationship between the other two parameters was analyzed. When $\mu_p$ is fixed, a strength spectrum with constant $\mu_p$ can be obtained. In this case, the analysis is relatively complex because iterative trial calculation is needed. An easier approach is to obtain a ductility spectrum with constant $R$. In this study, $R$ was fixed and deformation spectra with variable $\alpha$ and constant $R$ were obtained to investigate the $R-\mu_p-T-\alpha$ relationship. The strength reduction factor $R$ was set to 1.5, 2, 3, 4, 5, and 6. The initial period $T$ of the SDOF structure was varied from 0 to 6.0 s at a step length of 0.1 s. The post-yield stiffness ratio $\alpha$ was varied from 0 to 1 at a step length of 0.1. Figure 6 shows the seismic responses of a typical PYH SDOF system ($T = 0.3$ s, $R = 4$) under the action of Chi-Chi, CHY101, H1 ground motion. Clearly, the peak displacement and residual deformation of the PYH SDOF system with a certain post-yield stiffness are both less than those of the corresponding EPP structure.

![Figure 5. Load-displacement relationship of the SDOF system.](image)

![Figure 6. Comparison of the seismic responses of the EPP and PYH systems ($T = 0.3$ s, $R = 4$) under Chi-Chi, CHY101, H1 ground motion.](image)

Regarding the statistical analysis, the mean and standard deviation of the structural responses have been used to determine the $R-\mu_p-T$ relationship in past studies. It is not until recently that the mean and standard deviation of the logarithmic structural responses became increasingly popular, especially in research on probabilistic loss estimation because the lognormal distribution is deemed a better assumption for the seismic responses of structures. In this study, however, the traditional approach was used, that is, the mean and coefficient of variation (defined as the standard deviation divided by the mean) of the natural structural responses
were adopted as the statistical indicators. This is because, as was stated by Dolšek and Fajfar,\textsuperscript{47} it is conservative, familiar to users and used in most studies by other researchers.

3. PEAK DISPLACEMENT RESPONSES

3.1. Calculated results

As mentioned above, for the peak displacement response, the mean value $m(\mu_s)$ and the coefficient of variation $COV(\mu_s)$ were extracted for analysis. The coefficient of variation $COV(\mu_s)$ is defined as:

$$COV(\mu_s) = \frac{s(\mu_s)}{m(\mu_s)}$$  \hspace{1cm} (8)$$

where $s(\mu_s)$ is the standard deviation of $\mu_s$. The calculated results of the mean value $m(\mu_s)$ of the peak displacement are shown in Figure 7, and the following conclusions can be drawn. For short periods ($T < 1$ s), increasing the post-yield stiffness can greatly decrease the peak displacement of the structure; the shorter the period, the greater the decrease. For long periods ($T > 1$ s), increasing the post-yield stiffness can hardly reduce the peak displacement of the structure, approximately consistent with the equal-displacement criterion. The calculated results of $COV(\mu_s)$ are shown in Figure 8, and the following conclusions can be drawn. The increase in the post-yield stiffness can reduce the coefficient of variation, i.e., the discreteness of the peak displacement, especially for systems with periods smaller than 0.5 s. In addition, at a greater post-yield stiffness, the coefficient of variation tends to be approximately independent of the period. Furthermore, the coefficient of variation decreases as $R$ decreases, and $COV(\mu_s)$ should approach 0 when $R$ approaches 1.
Figure 7. Calculated and predicted $m(\mu_s)$ spectra.
3.2. Theoretical model

3.2.1. Mean value of $\mu_s$

As mentioned above, a conventional EPP structure can be determined by three parameters: $R$, $\mu_p$ and $T$. Over the past 40 years, many researchers have performed numerical and statistical analysis studies on the $R$-$\mu_p$-$T$ relation, and several predicted models have been proposed. However, for a PYH structure, a total of four parameters, $R$, $\mu_p$, $T$ and $\alpha$ should be considered. Determining the $R$-$\mu_p$-$T$-$\alpha$ relation directly using conventional numerical and statistical methods would be quite tedious; unfortunately, many of the existing research findings cannot be applied. Therefore, the method adopted in this study was used to determine the relationship between the seismic response of a PYH structure and corresponding elastic and EPP SDOF systems. The response of the elastic SDOF system can be obtained easily. In addition, the response of the EPP SDOF system can be obtained based on the existing research findings on the $R$-$\mu_p$-$T$ relation. Then, the response of the PYH structure can be conveniently predicted from the responses of the corresponding elastic and EPP structures.
Theoretically, for extremely short periods \((T \to 0\ s)\), the acceleration response of the structure is always equal to the peak acceleration of the input ground motion. Therefore, in this case, the equal-force criterion should be satisfied (Figure 9a):

\[ F_s = F_e \]  

(9)

In addition, according to the dynamic calculation results in Figure 7, for long periods \((T > 1\ s)\), the equal-displacement criterion should be satisfied (Figure 9c). Based on these two cases, a straight-line criterion for the PYH structure was proposed herein for any period. It means that, as shown Figure 9, the response point \((d_s, F_s)\) of the PYH structure is on the straight line determined by the response point \((d_e, F_e)\) of the corresponding elastic structure and the response point \((d_p, F_p)\) of the corresponding EPP structure. For extremely short periods \((T \to 0\ s)\), the straight-line criterion is equivalent to the equal-force criterion. For long periods \((T > 1\ s)\), the straight-line criterion is equivalent to the equal-displacement criterion. In general, \(F_s\) has a linear relationship with \(\mu_s\):

\[ F_s = \frac{F_e}{R} (m\mu_s + n) \]  

(10)
where \( m \) and \( n \) are the constant coefficients. Combining Equations (7) and (10), the following \( \alpha-\mu_s \) relation can be obtained:

\[
\alpha = \frac{m + n - 1}{\mu_s - 1} + m
\]

Figure 10 shows that \( \alpha \) is inversely proportional to \( (\mu_s - 1) \). The coefficients \( m \) and \( n \) can be solved based on the fact that the elastic response point \((R, 1)\) and the EPP response point \((\mu_p, 0)\) are both on the \( \alpha-\mu_s \) model curve. Then, Equation (11) can be written as

\[
\frac{\mu_s - 1}{\mu_s - 1} \frac{1}{R-1} = \alpha
\]

or

\[
\mu_s = \frac{1}{\alpha} + \frac{1}{\mu_s - 1} R-1
\]

Furthermore, the \( R-\mu_s-T \) relation developed by Ruiz-García and Miranda\(^{45}\) is

\[
\mu_s = R[1 + d(R - 1)]
\]

where

\[
d = \frac{1}{a(T/T_c)} \left[ \frac{1}{c} \right]
\]

and \( a, b \) and \( c \) are the constant coefficients and \( T_c \) is the characteristic period of the ground motion. The constant coefficients \( a = 50, b = 1.8 \) and \( c = 55 \) were proposed by Ruiz-García and Miranda.\(^{45}\) However, due to the different ground motion inputs and the differently used hysteretic behavior of systems, the constant coefficients \( a = 5, b = 2.4 \) and \( c = -50 \) were adopted in this study according to the fitting of the calculation results (Figure 11). Then, combining Equation (12-2) and Equation (13-1), the following \( R-\mu_s-T-\alpha \) relation can be obtained:

\[
\mu_s = \frac{1 + Rd}{1 + \alpha Rd} + 1
\]

where \( d \) is defined in Equation (13-2). Figure 7 compares the calculated \( m(\mu_s) \) spectra from the dynamic analysis and the predicted \( m(\mu_s) \) spectra from the theoretical model in Equation (14). The ratio of the predicted \( m(\mu_s) \) spectra to the calculated \( m(\mu_s) \) spectra is between approximately 0.8 and 1.2. Therefore, the theoretical model can estimate the numerical calculation results well. Figure 12 compares the straight-line criterion for the PYH structure and the Newmark criterion \(^{35}\) for the EPP structure. While the Newmark criterion can be used to determine the EPP response from the corresponding elastic response, the straight-line criterion can be used to determine the PYH response from the corresponding EPP response and elastic response.

### 3.2.2. Coefficient of variation of \( \mu_s \)

According to the calculation results shown in Figure 8, a theoretical model, described by a piecewise function of the period, was proposed here to predict the coefficient of variation of \( \mu_s \):

\[
COV(\mu_s) = \begin{cases} 
\frac{T}{T_s} + q_1 & 0 < T < T_s \\
\frac{T-T_s}{3-T_s} + q_2 & T_s \leq T < 3 s \\
\frac{T-3}{3} + q_3 & 3 \leq T \leq 6 s
\end{cases}
\]

where \( q_1, q_2, q_3 \) and \( q_4 \) are related to \( \alpha \) and \( R \):

\[
q_1 = 1.40(1 - \alpha^{0.2})(1 - 1/R)^{0.01} \\
q_2 = 1.25(1 - \alpha^{0.7})(1 - 1/R)^{0.00} \\
q_3 = 0.90(1 - \alpha^{1.2})(1 - 1/R)^{0.00} \\
q_4 = 0.28(1 - \alpha^{1.0})(1 - 1/R)^{0.00}
\]

and \( T_s \) is the characteristic period of the ground motion. Figure 8 compares the theoretical model and the calculation.
results of $COV(\mu)$. Therefore, the theoretical model can give an approximate estimation of the numerical calculation results.

(a) Short-period structures  
(b) Long-period structures

Figure 12. Comparison of the straight-line criterion and the Newmark criterion

3.3. Design response spectrum

Generally, the elastic pseudo-acceleration spectrum $S_{ae}$ is specified in the seismic design code. The elastic displacement spectrum $S_{de}$ can be expressed by $S_{ae}$:

$$S_{de} = \frac{S_{ae}}{T^2}$$  \hspace{1cm} (16)

Then, the acceleration spectrum $S_{ap}$ and displacement spectrum $S_{dp}$ of the EPP structure can be expressed by the elastic spectra and Equations (4) and (5).

$$S_{ap} = \frac{1}{R} S_{ae}$$ \hspace{1cm} (17-1)

$$S_{dp} = \frac{\mu_s}{R} S_{ae}$$ \hspace{1cm} (17-2)

Furthermore, the acceleration spectrum $S_{as}$ and displacement spectrum $S_{ds}$ of the PYH structure can also be expressed by the elastic spectra, following Figure 5.

$$S_{as} = S_{ae} \frac{\mu_s}{R}$$ \hspace{1cm} (18-1)

$$S_{ds} = S_{ae} \frac{1 + \alpha (\mu_s - 1)}{R}$$ \hspace{1cm} (18-2)

An example with a constant strength reduction factor ($R = 4$) is shown in Figure 13. The peak acceleration of the elastic spectrum is 1.0 g, the damping ratio is 0.05, and the characteristic period is 0.5 s. Similarly, the strength spectra with a constant deformation coefficient $\mu_s$ can also be given. $R$ can be solved by Equation (14):

$$R = \begin{cases} \frac{1}{2} \left( \frac{1}{d} - \alpha \mu_s + \alpha - 1 \right) + \frac{1}{2} \sqrt{\left( \frac{1}{d} - \alpha \mu_s + \alpha - 1 \right)^2 + \frac{4 \mu_s}{d}} & (d > 0) \\ \frac{1}{2} \left( \frac{1}{d} - \alpha \mu_s + \alpha - 1 \right) - \frac{1}{2} \sqrt{\left( \frac{1}{d} - \alpha \mu_s + \alpha - 1 \right)^2 + \frac{4 \mu_s}{d}} & (d < 0) \end{cases}$$ \hspace{1cm} (19)

An example with a constant deformation coefficient ($\mu_s = 4$) is shown in Figure 14.
Note that this study makes use of the existing research results of EPP structures to calculate the mean peak displacement responses of PYH structures. In addition to Equation (13), other published $R$-$\mu_p$-$T$ relations can also be used, changing Equations (14) and (19).

\[ \mu_p \text{ spectrum} \]  

(a) \[ \mu_p \text{ spectrum} \]  
(b) Acceleration spectrum

**Figure 13.** Design spectra of the peak responses with a constant strength reduction factor ($R = 4$).

\[ \mu_s \text{ spectrum} \]  

(a) $R_s$ spectrum  
(b) Acceleration spectrum

**Figure 14.** Design spectra of the peak responses with a constant ductility coefficient ($\mu_s = 4$).

### 3.4. Damage index

Various damage indexes for structural performance evaluation have been proposed by researchers. Among these, the damage index $DM$, based on the peak inelastic displacement without considering the cumulative hysteresis effect, is relatively simple and widely used.\(^{26}\)

\[ DM = \frac{\mu_u - 1}{\mu_u - 1} \]  

where $\mu_u$ represents the ultimate deformation capacity of a member or structure. As mentioned above, in long periods, the behaviors of the PYH structure and corresponding elastic and EPP structures are consistent with the equal-displacement criterion. According to the conventional damage index, if the peak displacements of the three structures are the same, the damage will be the same; however, this is not reasonable because the elastic structure must undergo no damage. The conventional damage index is appropriate for only the EPP structure and is not appropriate for the PYH and elastic structures. Therefore, the conventional damage index is expanded to be

\[ DM_s = (1 - \alpha) \frac{\mu_u - 1}{\mu_u - 1} \]  

This new index separates the elastic deformation and the EPP deformation of the PYH structure. For the EPP deformation, the conventional damage index is used. For the elastic deformation, no damage is considered. Therefore, this new index, $DM_s$, is appropriate for all three types of structures. Furthermore, the spectrum of the damage index $DM_s$ can also be plotted. Figure 15 gives an example of the design spectrum based on Figure 13a by assuming that $\mu_u = 8$. Clearly, $DM_s$ decreases with $\alpha$, especially for short periods.
4. RESIDUAL DEFORMATION RESPONSES

4.1. Calculated results

To investigate the residual deformation demands of SDOF systems, the residual deformation coefficients were defined by previous researchers by normalizing the residual deformation with respect to the peak inelastic displacement, peak elastic displacement, maximum possible residual deformation, or peak plastic deformation (the peak inelastic displacement minus the yield displacement). All these definitions have been evaluated in this study, and the last definition was adopted here because some rules were found to be convenient for mathematical fitting:

\[
\delta = \frac{d_r}{d_y} (\mu - 1)
\]

where \(d_r\) is the residual deformation and \(d_y\) is the yield displacement of the SDOF system. The mean value \(m(\delta)\) and the coefficient of variation \(\text{COV}(\delta)\) of the residual deformation coefficient \(\delta\) are extracted for analysis, and \(\text{COV}(\delta)\) is defined as

\[
\text{COV}(\delta) = \frac{s(\delta)}{m(\delta)}
\]

where \(s(\delta)\) is the standard deviation of the residual deformation coefficient \(\delta\). The calculated results of the \(m(\delta)\) spectra are shown in Figure 16, and the following conclusions can be drawn. With the increase in the post-yield stiffness, the residual deformation decreases rapidly. In addition, the shorter the period or the larger the \(R\), the more rapidly the residual deformation decreases. Additionally, when \(\alpha = 0\), \(m(\delta)\) nearly remains constant with a constant \(R\).

The calculated results of \(\text{COV}(\delta)\) are shown in Figure 17, and the following conclusions can be drawn. With a constant strength reduction factor \(R\), \(\text{COV}(\delta)\) can be approximated as a constant for any \(T\) and \(\alpha\). \(\text{COV}(\delta)\) decreases with \(R\). Theoretically, when \(R\) approaches 1, \(\text{COV}(\delta)\) should approach 0.

4.2. Theoretical model

4.2.1. Mean value of \(\delta\)

According to the calculation results shown in Figure 16, a theoretical model was proposed here to predict the mean residual deformation coefficient \(m(\delta)\):

\[
m(\delta) = (0.25 + \frac{0.06}{R - 0.8}) \times \frac{p^2}{1 + 2p} \times \left[\frac{1 + p}{\alpha + p}\right]^2
\]

\[
p = \frac{0.5T}{(T + 1)(R - 1)^{1.5}}
\]

Figure 16 also shows a comparison between the theoretical model and calculation results; the theoretical model can give a relatively good estimation of the numerical calculation results.

4.2.2. Coefficient of variation of \(\delta\)

According to the calculation results shown in Figure 17, a theoretical model assuming that \(\text{COV}(\delta)\) is related to only \(R\) was proposed here to predict the coefficient of variation of the residual deformation coefficient:

\[
\text{COV}(\delta) = \frac{s(\delta)}{m(\delta)}
\]
\[ COV(\delta) = \begin{cases} \left( \frac{R-1}{3} \right)^{0.15} & 1 \leq R < 4 \\ 1 & R \geq 4 \end{cases} \]  

Figure 17 shows a comparison between the theoretical model and calculation results; the theoretical model can give an approximate estimation of the numerical calculation results.

### 4.3. Design response spectrum

The mean residual deformation responses \( \frac{d_r}{d_y} \) can be solved from the mean residual deformation coefficient \( m(\delta) \) and the mean peak displacement response \( m(\mu) \) by combining Equations (14), (22) and (24-1):

\[
\frac{d_r}{d_y} = \left[ (R-1) \frac{1 + Rd}{1 + \alpha Rd} \right] \times \frac{0.25 + 0.06}{R - 0.8} \times \frac{p^2}{1 + 2p} \times \left( \frac{1 + p}{\alpha + p} \right)^{-1}
\]

(26)

where \( d \) is defined in Equation (13-2) and \( p \) is defined in Equation (24-2). Then, the spectrum of the mean residual deformation responses \( \frac{d_r}{d_y} \) can also be plotted. Figure 18 gives an example spectrum; in Figure 18b, for the EPP structure (\( \alpha = 0 \)), the laws and trends of the \( \frac{d_r}{d_y} \) spectrum coincide with the numerical results provided by Ruiz-Garcia and Miranda.14

Figure 16. Calculated and predicted \( m(\delta) \) spectra.
Figure 17. Calculated and predicted $COV(\delta)$ responses.
5. ILLUSTRATIVE EXAMPLE

To demonstrate the applicability of the proposed calculation method, an RC bridge column pier reinforced by a hybrid of steel and FRP bars has been analyzed. The peak drift and residual deformation were evaluated. The height of the column pier is 9 m. The lumped mass of the superstructure is $1.2 \times 10^5$ kg. The damping ratio is 0.05, and the characteristic period value is 0.5 s. The design peak acceleration response spectrum was set to $S_{DS} = 0.89$ g.

**Step 1:** A strength reduction factor is first assumed, such as $R = 3.3$.

**Step 2:** The yielding capacity demand of the column pier is $F_{yd} = 0.89 \times 9.8 \times 1.2 \times 10^5 = 317$ kN. Then, the design of the column pier can be carried out to ensure that the yielding capacity of the column pier is over $F_{yd} = 317$ kN. The cross section and reinforcement then can be determined.

**Step 3:** According to the cross section and reinforcement, the yield point and ultimate point of the pier can be determined. The following are assumed here: yield point (32 mm, 320 kN) and ultimate point (200 mm, 600 kN). Then, the period $T = 0.688$ s and post-yield stiffness ratio $\alpha = 0.167$ can be calculated.

**Step 4:** The peak inelastic displacement can be calculated using Equations (8), (15) and (16) and then compared with the allowable limit displacement of the pier structure.

The mean value and mean value plus one standard deviation of the peak inelastic displacement demand are $d_p = 3.987 \times 32 = 127.6$ mm and $d_p = (3.987 + 1.738) \times 32 = 183.2$ mm, respectively, which are both smaller than the limit of 200 mm.

**Step 5:** The residual deformation can be calculated using Equations (23) and (25) and then compared with the allowable values of the pier structure.

The mean value and mean value plus one standard deviation of the peak inelastic displacement demand are $d_p = 3.987 \times 32 = 127.6$ mm and $d_p = (3.987 + 1.738) \times 32 = 183.2$ mm, respectively, which are both smaller than the limit of 200 mm.
\[ COV(\delta) = \left( \frac{R - 1}{3} \right)^{0.15} = \frac{3.3 - 1}{3}^{0.15} = 0.96, \text{ and} \]
\[ s(\delta) = m(\delta) \times COV(\delta) = 0.017 \times 0.96 = 0.016. \]

The mean value and mean value plus one standard deviation of the residual deformation are \( d_r = 0.017 \times (3.987 - 1) \times 32 = 1.62 \text{ mm} = H / 5555 \) and \( d_r = (0.017 + 0.016) \times (3.987 - 1) \times 32 = 3.15 \text{ mm} = H / 2853 \) respectively, which are both smaller than the limit of \( H / 100 \) given by the Japanese seismic design specifications for highway bridges.  

*Step 6:* If Step 4 or 5 is not satisfied, return to Step 1 and adjust \( R \).

6. CONCLUSIONS

In this paper, the seismic responses of a PYH SDOF system with high-strength elastic materials were studied. Extensive dynamic time-history calculations were carried out for the peak displacement and residual deformation of the PYH SDOF structures, and the results were statistically analyzed. The main conclusions are drawn as follows.

1. The PYH behaviors of the materials, members and structures with high-strength elastic materials were observed in the experimental tests and summarized. The findings and conclusions of the previous studies on structures with EPP behavior or small post-yield stiffness may not be accurate for PYH structures.

2. In this study, the hysteresis model of a PYH SDOF system is assumed to be the combination of a conventional TKS model and a linear elastic model in parallel. Therefore, the findings of this research may not be appropriate for systems that exhibit a different behavior, even if they also employ high-strength elastic materials.

3. The mean values and coefficients of variation of the peak displacement were obtained and discussed. A new straight-line criterion was proposed to predict the mean peak displacement. A theoretical model for the coefficient of variation was also given. The inelastic response spectra based on the pseudo-acceleration response spectrum from the code for seismic design were presented. In addition, a new damage index was proposed for the evaluation of PYH structures.

4. The mean values and coefficients of variation of the residual deformation were obtained and discussed. Theoretical models for the seismic design or evaluation of PYH structures were proposed. The corresponding inelastic response spectra based on the pseudo-acceleration response spectrum from the seismic design code were also presented.

5. An RC bridge column pier was analyzed as an example. The peak drift and residual deformation were evaluated.

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