一道例题引发的“血”案

—— 向科学致敬

事情是这样的……

- W F Chen: Constitutive Equations for Engineering Materials, Example 3.7, pp680
- 在（σ,τ）应力子空间里，Prager法则与v. Mises准则合用时，屈服面除了平动，还会变形：

$$f=\frac{1}{3}\left(\sigma-\frac{2\sigma_a}{3}c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-\left(k-\frac{2\sigma_a{}^2}{9}c^2\mathrm{d}\lambda^2\right)=0$$

然而……

- 俺们咋也推不出wfChen的结果：

$$f=\frac{1}{3}\left(\sigma-\frac{2\sigma_a}{3}c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-\left(k-\frac{2\sigma_a{}^2}{9}c^2\mathrm{d}\lambda^2\right)=0$$

- 我们的结果：

$$f=\frac{1}{3}\left(\sigma-\sigma_a c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-k=0$$

  - 没有变形！
  - 平动比wfChen的结果要快！ ？

初次交锋……

- H.B. Liu

$$f=\frac{1}{3}\left(\sigma-\sigma_a c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-k=0 \quad \checkmark$$

But $\mathrm{d}\alpha_{ij}=c\cdot\mathrm{d}\varepsilon_{ij}{}^P=c\cdot\frac{\partial f}{\partial\sigma_{ij}}\mathrm{d}\lambda=c\cdot s_{ij}\cdot\mathrm{d}\lambda$

$$=c\mathrm{d}\lambda\begin{bmatrix}\frac{2}{3}\sigma_a & \tau_a & 0\\ \tau_a & -\frac{1}{3}\sigma_a & 0\\ 0 & 0 & -\frac{1}{3}\sigma_a\end{bmatrix}$$

初次交锋……

- H.B. Liu:

  如果后继屈服面里出现的不是Reduced Stress里的任何一个分量，而是别的什么形式，那后继屈服面就不仅是初始屈服面平动的结果，而包含了变形！

$$f=\frac{1}{3}\left(\sigma-\sigma_a c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-k=0$$

$$f=\frac{1}{3}\left(\sigma-\frac{2}{3}\sigma_a c\mathrm{d}\lambda-\frac{1}{3}\sigma_a c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-k=0$$

$$f=\frac{1}{3}\left(\sigma-\frac{2}{3}\sigma_a c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-(k-X)=0$$

Y. L. Wong: 这是“典型忽悠”！

节外生枝……

- 张量推导与矩阵推导的区别

  -1- Matrix form in stress subspace

$$f=\frac{1}{3}\sigma^2+\tau^2-k=0 \qquad [\mathrm{d}\alpha]=c[\mathrm{d}\varepsilon^p]=\begin{Bmatrix}\frac{\partial f}{\partial\sigma}\\ \frac{1}{2}\cdot\frac{\partial f}{\partial\tau}\end{Bmatrix}c\mathrm{d}\lambda=\begin{Bmatrix}\frac{2}{3}\sigma\\ \tau\end{Bmatrix}c\mathrm{d}\lambda$$

$$f=\frac{1}{3}\left(\sigma-\frac{2}{3}\sigma_a c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a\right)^2-k=0$$

应使用张量应变而非工程应变

$$\frac{\partial f}{\partial\tau}=d\varepsilon^p_{eng12}=2d\varepsilon^p_{12}$$

咋不一样哩？

  -2- Tensor form

$$f=\frac{1}{3}\left(\sigma-\sigma_a c\mathrm{d}\lambda\right)^2+\left(\tau-\tau_a c\mathrm{d}\lambda\right)^2-k=0$$

-3- Matrix form in complete stress space

$$f=\frac{1}{6}\left\{\begin{array}{l}\left[(\sigma_{11}-\alpha_{11})-(\sigma_{22}-\alpha_{22})\right]^2\\+\left[(\sigma_{22}-\alpha_{22})-(\sigma_{33}-\alpha_{33})\right]^2\\+\left[(\sigma_{11}-\alpha_{11})-(\sigma_{33}-\alpha_{33})\right]^2\end{array}\right\}\qquad \mathrm{d}\alpha_{ij}=c\mathrm{d}\lambda\begin{bmatrix}\frac{2}{3}\sigma_a & \tau_a & 0\\ \tau_a & -\frac{1}{3}\sigma_a & 0\\ 0 & 0 & -\frac{1}{3}\sigma_a\end{bmatrix}$$

$$+(\sigma_{12}-\alpha_{12})+(\sigma_{23}-\alpha_{23})+(\sigma_{13}-\alpha_{13})-k=0$$

$$f=\frac{1}{3}(\sigma-\alpha_{11}+\alpha_{22})^2+(\tau-\alpha_{12})^2-k=0$$

$$f=\frac{1}{3}(\sigma-\sigma_a c\mathrm{d}\lambda)^2+(\tau-\tau_a c\mathrm{d}\lambda)^2-k=0$$

与张量形式的推导一致，且不存在保持形式的问题

再次交锋……

- H.B. Liu:
  - 首先，后继屈服面在（σ,τ）应力子空间里肯定变形了！
  - 其次，后继屈服面在全应力空间中没有变形，但在主应力空间中发生了转动。
  - 然而，屈服面的转动是不允许的。所以**wfChen**说**Prager**法则会导致不一致的结果。
  - 至于是怎么变的，我也说不清楚。

又被忽悠回来了☹

求助祖师爷……

- Ziegler, H., [1959], "A Modification of Prager's Hardening Rule," Quarterly of Applied Mathematics 17, pp55-65
  - 使用 **v. Mises**屈服准则时，无论使用**Prager**还是**Ziegler**强化法则，结果都是一样的。即无论在平面应力还是平面应变条件下，都是标准的随动强化，屈服面不会发生变形！
  - 使用**Tresca**屈服准则时，**Prager**强化法则往往会导致屈服面的变形！

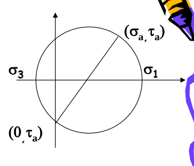
不幸的是，**w.f.Chen**偏偏选了**v. Mises**准则做例题

我们的推导是正确的☺

进一步探究……

- **Tresca + Prager**

$$\begin{bmatrix}\sigma_a & \tau_a & 0\\ \tau_a & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$



$$\sigma_1=\frac{\sigma_{11}+\sigma_{22}}{2}+\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2+\sigma_{12}{}^2}$$

$$\sigma_3=\frac{\sigma_{11}+\sigma_{22}}{2}-\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2+\sigma_{12}{}^2}$$

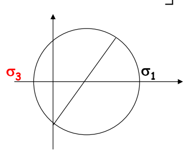
Tresca: $\left(\frac{\sigma_{11}+\sigma_{22}}{2}\right)+\sigma_{12}{}^2-k=0$

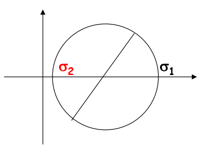
Tresca: $f=\left(\frac{\sigma_{11}+\sigma_{22}}{2}\right)+\sigma_{12}{}^2-k=0$

$$\mathrm{d}\alpha_{ij}=c\cdot\mathrm{d}\varepsilon_{ij}{}^P=c\cdot\frac{\partial f}{\partial\sigma_{ij}}\mathrm{d}\lambda=c\cdot\mathrm{d}\lambda\begin{bmatrix}\sigma_{11}-\sigma_{22} & \sigma_{12} & 0\\ \sigma_{12} & -(\sigma_{11}-\sigma_{22}) & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\overline{\sigma}_{ij}=\begin{bmatrix}\sigma-\sigma_a c\mathrm{d}\lambda & \tau_a & 0\\ \tau_a & \sigma_a c\mathrm{d}\lambda & 0\\ 0 & 0 & 0\end{bmatrix}$$

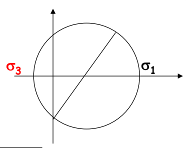
派生出两种可能





Case 1:

$$\overline{\sigma}_{ij}=\begin{bmatrix}\sigma-\sigma_a c\mathrm{d}\lambda & \tau_a & 0\\ \tau_a & \sigma_a c\mathrm{d}\lambda & 0\\ 0 & 0 & 0\end{bmatrix}$$



$$\sigma_1=\frac{\sigma}{2}+\sqrt{\left(\frac{\sigma-2\sigma_a c\mathrm{d}\lambda}{2}\right)^2+(\tau-\tau_a c\mathrm{d}\lambda)^2}$$

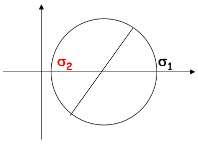
$$\sigma_3=\frac{\sigma}{2}-\sqrt{\left(\frac{\sigma-2\sigma_a c\mathrm{d}\lambda}{2}\right)^2+(\tau-\tau_a c\mathrm{d}\lambda)^2}$$

Tresca: $\overline{f}=\frac{1}{4}(\sigma-2\sigma_a c\mathrm{d}\lambda)^2+(\tau-\tau_a c\mathrm{d}\lambda)^2-k=0$

椭圆

Case 2:

$$\bar{\sigma}_{ij} = \begin{bmatrix} \sigma-\sigma_a \mathrm{c}d\lambda & \tau_a & 0 \\ \tau_a & \sigma_a \mathrm{c}d\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma-2\sigma_a \mathrm{c}d\lambda}{2}\right)^2 + \left(\tau-\tau_a \mathrm{c}d\lambda\right)^2}$$

$$\sigma_3 = 0$$

Tresca: $$\bar{f} = \frac{\frac{\sigma}{2} + \sqrt{\left(\frac{\sigma-2\sigma_a \mathrm{c}d\lambda}{2}\right)^2 + \left(\tau-\tau_a \mathrm{c}d\lambda\right)^2}}{2} - \sqrt{k} = 0$$

$$\left(2\sqrt{k}-\sigma_a \mathrm{c}d\lambda\right)\left[\sigma-\left(\sigma_a \mathrm{c}d\lambda+2\sqrt{k}\right)\right] = -\left(\tau-\tau_a \mathrm{c}d\lambda\right)^2$$

抛物线

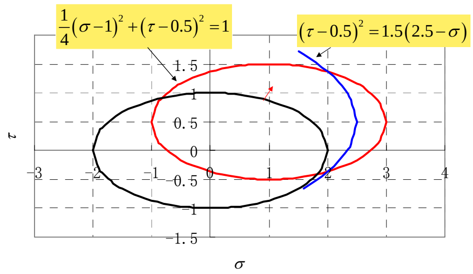
$$\frac{1}{4}\left(\sigma-2\sigma_a \mathrm{c}d\lambda\right)^2 + \left(\tau-\tau_a \mathrm{c}d\lambda\right)^2 = k$$

$$\left(\tau-\tau_a \mathrm{c}d\lambda\right)^2 = -\left(2\sqrt{k}-\sigma_a \mathrm{c}d\lambda\right)\left[\sigma-\left(\sigma_a \mathrm{c}d\lambda+2\sqrt{k}\right)\right]$$

set : $\tau_a \mathrm{c}d\lambda = 0.5$; $\sigma_a \mathrm{c}d\lambda = 0.5$; $k = 1$, then

$$\frac{1}{4}\left(\sigma-1\right)^2 + \left(\tau-0.5\right)^2 = 1$$

$$\left(\tau-0.5\right)^2 = 1.5\left(2.5-\sigma\right)$$



Subsequent yield surface deformed in a strange way, which is not allowed !