Proper interpretation of sectional analysis results

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Abstract: Displacement control algorithms commonly used to evaluate axial force-bending moment (PM) diagrams may lead to incorrect interpretations of the strength envelop for asymmetric sections. This paper aims to offer valuable insights by comparing existing displacement control algorithms with a newly proposed force control algorithm. The main focus is on the PM diagrams of three example sections that exhibit varying degrees of asymmetry. The comparative study indicates that conventional displacement control algorithms inevitably introduce non-zero out-of-plane bending moments. The reported PM diagram in such cases erroneously neglects the out-of-plane moment and fails to represent the strength envelope accurately. This oversight results in significant and unconservative errors when verifying the strength of asymmetric sections.

Keywords: fiber section model, axial force-moment interaction diagram, limit state, failure surface

1. Introduction

Reinforced concrete and composite structural elements are commonly used in building and bridge construction and are typically subject to the combined effects of biaxial bending and axial load. It is essential to assess the adequacy of these sections at the ultimate limit state (via axial force-moment interaction diagrams, usually referred to as PM curves or PMM curves) or provide information on their inelastic response.
gradually up to failure (in the form of moment-curvature curves) to structural seismic
design. The analysis of composite sections under biaxial bending and axial load has
received significant attention in the literature (Furlong, 1961; Penelis, 1969; Werner,
1974). Many commonly used sections do not fall into the regular/symmetric category.
Such sections include but are not limited to L-shaped columns/wall piers, symmetric
sections with asymmetrically placed structural steel and reinforcement, repaired
sections, and various composite sections (Tsao and Hsu, 1993; EI-Tawil et al., 1995;
Hajjar and Gourley, 1996; Liang et al., 2006).

In the realm of engineering structures, the advent of powerful and cost-effective
personal computers has facilitated the analysis of reinforced concrete (RC) columns
and composite sections subjected to biaxial bending and axial load through the
application of the "fiber" section model (Warner, 1969; Lachance, 1980; Al-Noury and
Chen, 1982; Dinsmore, 1982; Tsao and Hsu, 1993; EI-Tawil et al., 1995; Spacone et al.,
1996a, 1996b). This approach has proven to be highly effective, producing results
demonstrating remarkable agreement with experimental data for flexural failure modes
and monotonic loading, rendering it a commonly employed tool in structural design. Its
capability of efficiently generating moment-curvature curves and axial force-moment
interaction diagrams further bolstered its utility in engineering structures.

The fiber section model has a simple concept that involves dividing a section into
a series of $n$ fibers, which do not necessarily have the same area, and integrating the
stresses over the cross-sectional area to obtain stress resultants, including force or
moment (EI-Tawil et al., 1995; Spacone and EI-Tawil, 2004). Each fiber can be
assigned different material properties such as concrete, structural steel, or reinforcing bars, and the 'plane sections remain plane' assumption is utilized to calculate fiber stresses from the fiber strain using relevant material constitutive models. The moment-curvature results are evaluated after mapping the materials to the sectional fibers and applying an axial load.

With advancements in cross-sectional analysis technology, engineers have access to various cross-sectional analysis software or programs that can be used to simplify this process (Mahin and Bertero, 1977; Bentz, 2000; Chadwell, 2004; Charalampakis and Koumousis, 2008; Papanikolaou, 2012). XTRACT (Chadwell and Imbsen, 2004) is one of the most widely used tools for various tasks, including moment-curvature analysis, axial force-moment interaction analysis, and axial force-ultimate curvature analysis. CiSDesigner (2015) is another prevalent tool in China.

Two general categories of methods are used in such sectional analyses: displacement control and force control (Chadwell, 2004). Displacement control is a common approach in cross-sectional analysis programs due to its direct and robust nature (Charalampakis and Koumousis, 2008). Both XTRACT and CiSDesigner adopt the displacement control method. This method involves keeping a fixed orientation of the neutral axis and incrementing the curvature of the section until a specified limit state is reached. For asymmetrical cross sections, a secondary bending moment is necessary to establish the deformed state of the section (Sfakianakis, 2002). Consequently, the points on the failure surface follow a complex non-planar 3-dimensional (3D) curve. It can be anticipated that this type of 3D curve poses big challenges for engineers.
attempting to plot and use it for seismic design.

In force control algorithms, on the other hand, the forces (or moments) are incremented until a limit state is reached. In this process, we need to calculate the angle and depth of the neutral axis of the section through the force acting on it, to confirm the strain state of the section, and these are nonlinear equations with several parameters all coupled together, requiring an iterative approach such as the quasi-Newton method suggested by Yen (1991). In a force control solution strategy, the inverse problem is solved. Therefore, through force control, the designer can determine the interaction curves for each particular moment ratio or $\theta$ (i.e., $M_{nx}/M_{ny} = \tan \theta$) (Rodriguez and Aristizabal-Ochoa, 1999). This technique generates plane interaction curves for columns under biaxial bending, which are much easier to plot and use in actual design. However, these algorithms are not straightforward to implement, and, in some cases, they are sensitive to the selection of the origin of the reference loading axes. As a result, they may become unstable near the state of pure compression (Yen, 1991; Brondum-Nielsen, 1985). Meanwhile, with this type of solution, problems with convergence can occur when severe discontinuities are defined within the material models (Chadwell and Imbsen, 2004). In an asymmetric section, the center of rigidity does not coincide with the centroid, resulting in the axial force acting on the rigid center rather than on the centroid when the entire section reaches the ultimate compressive strain. Consequently, the PM curves at the point of maximum axial force exhibit a non-zero bending moment (Qu et al., 2014). This leads to situations in the section force control with fixed axial force, where different bending moments correspond to the same axial
force in the vicinity of the maximum axial force. Thus, it is difficult for such analytical methods to obtain correct results in this case.

This paper presents a novel algorithm for force-controlled sectional analysis characterized by efficient analysis, accurate calculation, and stable convergence. It enables a direct comparison of commonly used displacement control algorithms and the force control algorithm in generating PM and PMM curves. In this paper, the comparison is made on three example cross-sections with different degrees of asymmetry to highlight the problem of possible improper interpretation of a two-dimensional PM curve generated by a displacement control algorithm, to explain its causes, and to justify the superiority of the proposed force control algorithm in terms of avoiding such improper interpretation.

2 Methods of sectional analysis

The strain at the centroid of a section is often conveniently adopted as the axial strain of the section to avoid the eccentricity induced by an applied axial force. Displacement control prescribes curvatures ($\phi_x$, $\phi_y$) about the $x$ and $y$ axes of the section and determines the axial strain by trial and error. The ratio of the curvatures, $\phi_x/\phi_y$, is determined before the analysis by the initial or elastic bending stiffness of the section. For example, $\phi_x/\phi_y = 0$ for bending about the $y$ axes, and $\phi_x/\phi_y = 1$ for equally biaxial bending. With knowledge of the location of individual fibers, the fiber strain can be determined based on the assumption that plane sections remain plane. By utilizing stress-strain relations from material models, the fiber stress can be derived, followed by the calculation of the force ($F_i$) in each fiber via multiplication of the fiber
stress with its respective area. By force equilibrium, the overall axial force \( (P) \) and corresponding moments about the \( x \) and \( y \) axes \( (M_x \) and \( M_y) \) can be determined as:

\[
P = \sum_{i=1}^{N} F_i \quad M_x = \sum_{i=1}^{N} y_i F_i \quad M_y = \sum_{i=1}^{N} x_i F_i
\]

However, the total axial force found from summing the individual forces within each fiber is not necessarily equal to the applied axial load on the section, necessitating iteration between the applied axial load and the resisting axial force \( (P) \) via changes in the strain at the centroid. Upon achieving a match within a defined tolerance, the curvature is incremented, and the process is repeated. The incrementing curvatures terminate when a desired limit state is reached within the material. For reinforced concrete (RC) cross sections, the ultimate limit state is defined as the curvature at which the longitudinal reinforcement fracture or the confined concrete crushing occurs, as determined by the respective material models.

On the other hand, the proposed force-controlled algorithm prescribes the ratio of axial force and bending moments \( (p, m_x, m_y) \) such that

\[
P = \lambda p \quad M_x = \lambda m_x \quad M_y = \lambda m_y
\]

where \( \lambda \) is a scalar that varies during loading and unloading (Fig. 1).

Fig. 1. The analysis characteristics of the proposed force-controlled algorithm.

Force control is advantageous in that it captures the exact ultimate limit state \( (P_u, \)
\( M_{ux}, M_{uy} \) while ensuring the proportionality

\[
\frac{p_u}{p} = \frac{M_{ux}}{m_x} = \frac{M_{uy}}{m_y}
\]  

We adopt the multiple-point-constraint method (Huang, 2009; Huang et al., 2011) to solve the challenging requirement of force control. The robustness of the method was demonstrated by analyses applying proportional load in biaxial compression, pure shear, and pushover load cases. The constraint we impose specifically for the section analysis is given by

\[
(p \epsilon + m_x \phi_x + m_y \phi_y) - (p + m_x + m_y) d = 0
\]  

where \( d \) is a scalar degree of freedom (DOF) that controls the progression of the loading-unloading process.

A proof for the multi-point constraint in Eq. 4 to ensure the load proportionality is provided in the Appendix. Note that the displacement control varies axial strain, whereas force control varies DOF \( d \) in their iteration. Force control requires similar computational expense but offers profound benefits in ensuring load proportionality, as demonstrated in the following sections, where the sectional analyses results by the proposed force control method are compared with those by XTRACT and CiSDesigner for three example sections.

3 Example sections

As shown in Fig. 2, we conceived three example cross-sections of reinforced concrete (RC) members, which are named Section A (Fig. 2a), Section B (Fig. 2b), and Section C (Fig. 2c), for evaluating the different sectional analysis algorithms. The dimension of Section A is 400mm x 400mm, and the thickness of the concrete cover is
28mm. Its longitudinal rebars, whose diameter is 14mm, distribute uniformly around its perimeter, so Section A is a symmetric cross section. Section B has the exact dimensions as section A, but the diameter of steel bars on both sides is intentionally skewed to create a heavily asymmetric, though unrealistic, section. In particular, it changes the diameter of the leftmost rebars to 50 mm based on section A. In Section B, the thickness of the leftmost concrete cover is 17mm, while the thickness of the other concrete cover is 28mm. Section C is a 1600mm×1200mm×200mm L-shaped cross-section, with the thickness of the concrete cover is 25mm. Section C adopts 12-mm-diameter rebars whose arrangement is shown in Fig. 2c. Section C is a biaxially asymmetric section.

![Figure 2. Example RC cross sections: (a) symmetric column section, (b) asymmetric column section, and (c) L-shaped wall section. (unit: mm)](image)

The material properties of the concrete and rebars are summarized in Table 1, where $E_c$ and $E_s$ is the elastic modulus of concrete and reinforcement, respectively; $f_c$ is the compressive strength of concrete, and $f_y$ is the yield strength of reinforcement; $\varepsilon_{co}$ is the compressive strain related to the peak compressive strength $f_c$; $\varepsilon_{cu}$ is the crushing strain of concrete, and $\varepsilon_{su}$ is the fracture strain of
reinforcement. $f_t$ is the tensile strength of concrete.

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_c(E_s)$ /MPa</th>
<th>$f_c(f_y)$ /MPa</th>
<th>$\varepsilon_{co}$</th>
<th>$\varepsilon_{cu}(\varepsilon_{su})$</th>
<th>$f_t$ /MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>$3 \times 10^4$</td>
<td>20.1</td>
<td>0.002</td>
<td>0.003</td>
<td>2.01</td>
</tr>
<tr>
<td>Rebar</td>
<td>$2 \times 10^5$</td>
<td>400</td>
<td>-</td>
<td>0.1</td>
<td>-</td>
</tr>
</tbody>
</table>

4 PM curves and PMM surface

4.1 Material models and parameters

For the concrete, both the proposed algorithm and XTRACT adopt Mander's model (Mander et al., 1988) for the stress-strain relationship in compression (Fig. 3a). It should be noted that, after $\varepsilon_{cu}$, the descending branch ended at failure strain 0.005 is simplified as linear. CiSDesigner adopts Rüsch's model (Rüsch, 1960), which is represented by a combination of a parabola for the ascending part that ends at $\varepsilon_{co}$ and a straight line for the horizontal part that ends at $\varepsilon_{cu}$. In tension, the ascending and descending part of the stress-strain curve in the proposed algorithm and XTRACT are both linear, of which the absolute values of the slopes are both $E_c$, while CiSDesigner assumes a negligible tensile strength of concrete.

For the steel rebars, an elastic-perfectly plastic stress-strain relationship in both tension and compression is assumed by all the three software packages investigated in this study (Fig. 3b). The fracture strain of rebars $\varepsilon_{su}$ is set to be 0.1.
Figure 3. Stress-Strain Model. (a) stress-strain curve of concrete; (b) stress-strain curve of reinforcement.

4.2 Two-dimensional axial force-moment interaction curves

Fig. 4 compares the axial force $P$-moment $M_x$ interaction curves of the three example sections when they reach the ultimate limit state. The curves are calculated by the proposed algorithm, XTRACT, and CiSDesigner, where the positive axial forces are tensile, and the negative axial forces are compressive. The bending is about the $x$-axis of the example sections shown in Fig. 2. For the symmetric Section A (Fig. 4a), the analysis results of the proposed algorithm, XTRACT, and CiSDesigner are almost identical despite the different categories of solution algorithms and the slight difference in their adopted material models. For the asymmetric Sections B and C, however, the displacement-control-based results of XTRACT and CiSDesigner are similar but significantly different from the force-control-based results by the proposed algorithm (Fig. 4b and c).
Figure 4. Comparison of PM curves by the proposed algorithm, XTRACT, CiSDesigner for (a) Section A; (b) Section B; and (c) Section C.

In particular, the $P-M_x$ curve by the force control method demonstrates much smaller ultimate strengths of the section than those by displacement-control results by XTRACT and CiSDesigner. For Section B, the ultimate strengths under pure tension and compression are 3634kN and 6667kN by the displacement-control methods but are only 814.1kN and 4033kN by the force-control method. For Section C, the maximum flexural strength under compression is 4660kNm and is reached at a compression force of 704.2kN as estimated by XTRACT, but is only 3566kNm at 2868kN compression force by the force-control algorithm. If the force control results are assumed correct, there must be something wrong with the displacement control results, or vice versa. In the following sections, we will further explore the reason for such a disagreement.

In addition to the ultimate limit state, the proposed force-control algorithm can yield the PM curves at other limit states with superior stability. For example, Fig. 5 depicts the PM curves for the three example sections at three different limit states corresponding to concrete cracking, rebar yielding, and ultimate failure.
Figure 5. P-M curves by the proposed force control algorithm at three different limit states for (a) Section A; (b) Section B; and (c) Section C.

4.3 Three-dimensional axial force-moment interaction curves

Considering the influence of the moment about the y-axis, $M_y$, the results are re-plotted in the three-dimensional coordinate system in Fig. 6. For the symmetric Section A (Fig. 6a), the PMM curves by the three programs are almost identical. The PMM curve reduces to a two-dimensional one lying in the $xz$-plane, and its projection in the $yz$-plane becomes a straight line along the $y$-axis, that is, $M_y = 0$. In this case, the bending action about the $x$-axis does not introduce a secondary moment about the $y$-axis because the constituents within the symmetrical section (such as steel bars and concrete) are symmetrically arranged.
Figure 6. Comparison of PMM curve by the proposed algorithm, XTRACT, CiSDesigner for (a) Section A; (b) Section B; and (c) Section C.

For the uniaxially asymmetric Section B (Fig. 6b), the PMM curve by the proposed force-control algorithm exactly lies in the \(xz\)-plane, and its projection on the \(yz\)-plane remains a vertical straight line indicating a zero moment about \(y\)-axis, i.e., \(M_y = 0\). which means the proposed algorithm successfully ensures that. On the other hand, the PMM curves by XTRACT and CiSDesigner are deflected and do not lie in the \(xz\)-plane anymore but remain perpendicular to the \(yz\)-plane with their projection onto the \(yz\)-plane as inclined curves. In this case, the moment about \(y\)-axis \(M_x\) is not zeroed, i.e., \(M_y \neq 0\). For example, the aforementioned 3630kN 'pure' tensile strength by the displacement control is, in fact, not obtained in a pure tension state but is accompanied
by a $M_y=457.4\text{kNm}$ moment about the $y$-axis. Since it is not shown in a planar PM curve like that in Fig. 4b, this secondary moment $M_y$ maybe conveniently neglected, leading to an overestimate of the pure tensile strength of Section B.

For the biaxial asymmetric Section C (Fig. 6c), the PMM curve by the proposed force-control algorithm still remains in the $xz$-plane, indicating that the constraint of $M_y = 0$ is successfully achieved. On the other hand, the PMM curves by displacement-controlled XTRACT and CiSDesigner not only stretch out of the $xz$-plane, but also deflect around the $z$-axis. Its projection onto the $yz$-plane forms a closed loop. For example, the 4660kNm ultimate flexural strength at 704.2kN compressive force given by XTRACT is accompanied by a secondary moment $M_y = 1761\text{kNm}$. It explains the discrepancy from the force-controlled result of 3566kNm ultimate strength at 2868kN compressive force, which is obtained under a strict force boundary condition of $M_y=0$.

4.4 PMM Surfaces

The appropriateness is further studied by inspecting the PMM surface in the $xyz$-space. Fig.7 shows that the results by different methods agree with one another. The PMM surface aggregates a family of PMM curves obtained under various boundary conditions defined by forces or displacements.
Figure 7. Comparison of PMM surfaces by the proposed algorithm, XTRACT, CiSDesigner for: (a) Section A; (b) Section B; and (c) Section C.

The significant difference in the PM and PMM curves, as shown in the previous sections, arises from the different definitions of the boundary conditions when calculating a specific curve. It reminds us of the importance of properly acknowledging the boundary condition of the analysis. For asymmetric sections, the sectional analysis by displacement control method will inevitably introduce a non-zero secondary moment out of the loading direction, leading to a three-dimensional PMM curve. Its simple projection onto a P-M plane gives a two-dimensional PM curve that may give misleading and over-estimated strength. Therefore, in the case of analyzing asymmetrical sections using the displacement control algorithm, it is advisable to
employ the PMM surface to acknowledge the complete boundary conditions.

5 Conclusions

This study proposes a novel force-control algorithm for analyzing arbitrary sections under biaxial bending and axial load. It enables direct comparison with the commonly adopted displacement-control method in commercially available software packages like XTRACT and CiSDesigner. We can draw the following conclusions from comparing the axial force-moment interaction diagrams of the three example sections with different degrees of asymmetry obtained by the proposed force-control method and those by the displacement-control method.

(1) All three algorithms give consistent PMM surfaces despite their different solution strategies and slight differences in material models. This applies to all three example sections, even if the asymmetry is significant.

(2) Projecting a three-dimensional PMM curve obtained by a displacement control method onto a PM plane for asymmetric sections is inappropriate. The displacement control method inevitably introduces a secondary moment transverse to the loading direction. Such a projected PM curve overestimates the sectional strengths.

(3) The proposed force control algorithm can keep the force boundary condition prescribed by the force increments $m_x$, $m_y$, and $p$ while remaining stable and robust. It can also output PM and PMM curves at any limit states.

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Appendix

The kinetic constraint in Eq. A1 is proposed to ensure proportional loading described in Eq. 3.

\[(p \epsilon + m_x \phi_x + m_y \phi_y) - (p + m_x + m_y)d = 0 \quad (A1)\]

where \(p\), \(m_x\), and \(m_y\) are the load proportions, \(\epsilon\), \(\phi_x\) and \(\phi_y\) are the corresponding displacements, while \(d\) is an average weighted by the load proportions:

\[d = \frac{p \epsilon + m_x \phi_x + m_y \phi_y}{p + m_x + m_y} \quad (A2)\]

Application of the principle of virtual work to the rigid constraint leads to

\[F \delta d + (-p \delta \epsilon - M_x \delta \phi_x - M_y \delta \phi_y) = 0 \quad (A3)\]

where \(P\), \(M_x\), and \(M_y\) are the applied loads and \(F\) is a load conjugate to \(d\). Because the constraint is rigid, the internal virtual work vanishes at the right-hand side of Eq. A3. The corresponding virtual displacements, \(\delta d\) and \(\delta \epsilon\), \(\delta \phi_x\), \(\delta \phi_y\) are also required to satisfy the kinetic constraint in Eq. A1.

\[(p \delta \epsilon + m_x \delta \phi_x + m_y \delta \phi_y) - (p + m_x + m_y)\delta d = 0 \quad (A4)\]

Combining Eqs. A3 and A4 leads to

\[\left[ pF - (p + m_x + m_y)P \right] \delta \epsilon + \left[ m_x F - (p + m_x + m_y)M_x \right] \delta \phi_x + \left[ m_y F - (p + m_x + m_y)M_y \right] \delta \phi_y = 0 \quad (A5)\]

This derivation is valid even if \(d\) is under displacement control such that \(\delta d = 0\). As long as \(\epsilon\), \(\phi_x\) and \(\phi_y\) remain free degrees of freedom, Eq. A5 shall be satisfied for arbitrary variations \(\delta \epsilon\), \(\delta \phi_x\), and \(\delta \phi_y\). Thus, their coefficients vanish:

\[pF - (p + m_x + m_y)P = 0 \quad (A6a)\]
\[ m_x F - (p + m_x + m_y) M_x = 0 \quad \text{(A6c)} \]
\[ m_y F - (p + m_x + m_y) M_y = 0 \quad \text{(A6d)} \]

or more usefully

\[ P: M_x: Y = p: m_x: m_y \quad \text{if } p + m_x + m_y \neq 0 \quad \text{(A7)} \]

In other words, the constraint \( (p \epsilon + m_x \phi_x + m_y \phi_y) - (p + m_x + m_y) d = 0 \)

enforces load proportionality \( P: M_x: M_y = p: m_x: m_y \). The validity of the proposed

method is proved.

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Conflict of interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled “Proper interpretation of sectional analysis results”. The corresponding author Zhe Qu is an editorial board member for the Journal of Earthquake Research Advances and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.